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# A Comprehensive Analysis of the Fourth Hankel Determinant Associated with the Class of Janowski Starlike Functions with Complex Parameters

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**Abstract**

We consider the class of normalized holomorphic functions $f$ in such a way that the ratio $((f'(z))/(q(z))$ is subordinate to function $\frac{1+Az}{1+Bz}$ (Janowski function), and $q(z)$ is subordinate to $\frac{2}{1+e^{-z}}$ (sigmoid function), and we derive a fourth Hankel determinant bound for this class.

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## 1. Introduction

Let $\mathcal{H}$ be the family of holomorphic (or analytic) functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{N}_n \subset \mathcal{H}$ such that $f \in \mathcal{N}_n$ has the series representation

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j. \tag{1.1}$$

Let $\mathcal{S}$ be the subfamily of $\mathcal{N}_1 \equiv \mathcal{N}$ containing univalent functions in $\mathbb{U}$. Assume that $\mathbb{U}$ contains two analytic functions, $f$ and $g$. Then the function $f$ is therefore said to be subordinate to the function $g$, and the following expression could be written:

$$f(z) \prec g(z), \qquad (z \in \mathbb{U}). \tag{1.2}$$

If a Schwarz function $w(z)$ exists that meets the following criteria,

$$w(0) = 0 \text{ and } |w(z)| < 1, \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)), \quad (z \in U).$$

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Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic and regular in $\mathbb{U}$ with $p(0) = 1$, $\{Re p(z) > 0$, and also meets the following criteria

$$p(z) \prec (1 + Az/1 + Bz), \quad -1 \leq B < A, \ -1 < A < 1.$$

Then this function is referred to as a Janowski function and is represented by the letter $\mathcal{P}(A, B)$. Geometrically $p(z) \in \mathcal{P}(A, B) \Leftrightarrow p(0) = 1$ and $p(\mathbb{U})$ is inside the domain Specified by

$$\Omega(A, B) = \left\{ \omega : \left| \omega - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\},$$

with diameter end points

$$\frac{1 - A}{1 - B} = p(-1) \text{ and } \frac{1 + A}{1 + B} = p(1).$$

Let $\mathcal{S}^*(A, B)$ be the class of functions $\kappa(z)$, $\kappa(0) = 0 = \kappa'(0) - 1$ are holomor- phic in $\mathbb{m}_d$ and meets the following requirements

$$\kappa(z) \in \mathcal{S}^*(A, B) \Leftrightarrow \frac{z\kappa'(z)}{\kappa(z)} \prec \mathcal{P}(A, B).$$

In geometric function theory, the most basic and important subfamilies are the star- like, convex, and close-to-convex functions which are defined by

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \Re\left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{U} \right\},$$

$$\mathcal{C} = \left\{ f \in \mathcal{S} : \Re\left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{U} \right\},$$

$$\mathcal{k} = \left\{ f \in \mathcal{S} : \Re\left( \frac{zf'(z)}{g(z)} \right) > 0 \quad g(z) \in \mathcal{S}^* \ z \in \mathbb{U} \right\}.$$

Many scholars have created and researched distinct subclasses of analytic func- tions connected with various image domains, see the work of Cho et al. [31], Dziok et al. [28], Kumar and Ravichandran [29], Mediratta et al. [30], Soko-l and Stankiewicz [33], Raina and Sokol [35], Kanas and Raducanu [34], Sharma et al. [36]. Using the above-mentioned concepts, we now consider the following class:

$$q^* = \left\{ f \in \mathcal{N} : \frac{f'(z)}{q(z)} \prec \frac{1 + Az}{1 + Bz} \quad \& \quad q(z) \prec \frac{2}{1 + e^{-z}} \ z \in \mathbb{U} \right\}. \tag{1.3}$$

The $q$-th Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions $f$ was defined by Noonan and Thomas in 1976 as follows:

$$\Delta_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \quad (a_1 = 1).$$

The first, second, third and fourth-order Hankel determinants are provided here, in that sequence.

$$\Delta_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, (a_1 = 1,\ n = 1,\ q = 2,) \tag{1.4}$$

$$\Delta_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}, \ (n = 2,\ q = 2,) \tag{1.5}$$

$$\Delta_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \ (n = 1,\ q = 3,) \tag{1.6}$$

$$\Delta_{4,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}, \ (n = 1,\ q = 4). \tag{1.7}$$

Since $f \in S$ and $a_1 = 1$ thus

$$\begin{aligned} \Delta_{4,1}(f) = {} & a_7 \{a_3 I_1 - a_4 I_2 + a_5 I_3\} \\ & -a_6 \{a_3 I_4 - a_4 I_5 + a_6 I_3\} \\ & +a_5 \{a_3 I_6 - a_5 I_5 + a_6 I_2\} \\ & -a_4 \{a_4 I_6 - a_5 I_4 + a_6 I_2\}, \end{aligned}$$

where

$$I_1 = a_2 a_4 - a_3^2,\ I_2 = a_4 - a_2 a_3,\ I_3 = a_3 - a_2^2,$$
$$I_4 = a_2 a_5 - a_3 a_4,\ I_5 = a_5 - a_2 a_4,\ I_6 = a_3 a_5 - a_4^2.$$

There are few findings in the literature about the Hankel determinant for functions belonging to the general family S. The best proven sharp inequality for the function $f \in S$ is

$$|\Delta_{2,n}(f)| \le \mu \sqrt{n},$$

where $\mu$ is the absolute constant, specified by Hayman [37]. Furthermore, for the same class S, it was demonstrated in [38] that

$$|\Delta_{2,2}(f)| \le \mu \qquad \text{for } \mu \in \left[1, \frac{11}{3}\right] \tag{1.8}$$

$$|\Delta_{3,1}(f)| \le \lambda \qquad \text{for } \lambda \in \left[\frac{4}{9}, \frac{32 + \sqrt{85}}{15}\right]. \tag{1.9}$$

Extensive research has been conducted on the expansion of $|\Delta_{q,n}(f)|$ for various subfamilies of the set S of univalent functions. For example Janteng [39], [40] measured the sharp bound of $|\Delta_{2,2}(f)|$ for the subfamilies C, $S^*$, and k of the set S. These bounds are

$$|\Delta_{2,2}(f)| \le \begin{cases} 1 & f \in S^*, \\ \frac{1}{8} & f \in C, \\ \frac{4}{9} & f \in k. \end{cases} \tag{1.10}$$

The exact bound for the collection of close-to-convex functions with such a specified determinant is still unknown. Krishna and RamReddy [41] on the other hand, showed the best estimate of $|\Delta_{2,2}(f)|$ for the set of Bazilevic functions. The calculations in (1.6) make it clear that estimating $|\Delta_{3,1}(f)|$ is significantly more difficult than estimating the $|\Delta_{2,2}(f)|$ bound. Babalola [42] found the up- per bound of $|\Delta_{3,1}(f)|$ for the families of C, $\mathfrak{S}$, and k in the first publication on $|\Delta_{3,1}(f)|$, published in 2010. He came up with the following bounds:

$$|\Delta_{3,1}(f)| \leq \begin{cases} 16 & f \in S^*, \\ 0.174 & f \in C, \\ 0.742 & f \in k. \end{cases} \tag{1.11}$$

Other authors later used the same methods to publish their work on $|\Delta_{3,1}(f)|$ for other subfamilies of analytic and univalent functions. Zaprawa [43] improved Babalola's [42] results in 2017 by using a new technique described as

$$|\Delta_{3,1}(f)| \leq \begin{cases} \frac{49}{540} & f \in S^*, \\ 1 & f \in C, \\ \frac{41}{60} & f \in k. \end{cases}$$

He claims that such bounds are not the best ones. In 2018, the reader received the first studied manuscripts in which the authors obtained the sharp bounds of $|\Delta_{3,1}(f)|$. Kowalczyk et al. [44] and Lecko et al. [45] have written such works. These results are as follows:

$$|\Delta_{3,1}(f)| \leq \begin{cases} \frac{1}{9}, & f \in S^*\left(\frac{1}{2}\right), \\ \frac{4}{35} & f \in C, \end{cases}$$

where $S^*\left(\frac{1}{2}\right)$ symbolizes the starlike function family of order $\frac{1}{2}$. For functions with bounded turning, Arif et al. [1] and [2] determined the bounds on the fourth- and fifth-order Hankel determinants. Cho and Kumar [47] investigated the bound on $H_4(1)$ for the lune function class. Also Arif [46] just determined the fourth Hankel determinant bound for the class of star-like functions linked to the Bernoulli lemniscate. Many articles have been published in the last few years looking for upper bounds for the second-order Hankel determinant $H_2(2)$, third- order Hankel determinant $H_3(1)$ and fourth Hankel determinant $H_4(1)$, see for example [25, 14, 26]. We contributed to the field by identifying the fourth Hankel determinant for the class $q^*$. We consider the class $q^*$ of normalized holomorphic functions $f$ in such a way that

$$((f'(z))/(q(z)) \prec \frac{1+Az}{1+Bz} \quad \text{and } q(z) \prec \frac{2}{1+e^{-z}}.$$

where the notion "$\prec$" denotes the familiar subordinations and we derive a fourth Hankel determinant bound for this class.

**Main Results**

In order to prove our desired results, we shall require the following lemmas.

**Lemma 1.1.** *(see [13]) If $p(z) \in \mathcal{P}$, then exists some $x, z$ with $|x| \leq 1$, $|z| \leq 1$, such that*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x,$$

$$4c_3 = c_1^3 + 2c_1x\left(4 - c_1^2\right) - c_1x^2\left(4 - c_1^2\right) + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z.$$

**Lemma 1.2.** *(see [20]) Let $p(z) \in \mathrm{P}$. Then*

$$\left|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4\right| \le 2,$$
$$\left|c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5\right| \le 2,$$
$$\left|\begin{array}{c} c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 \\ -3c_1^2c_4 - 6c_1c_2c_3 - c_6 \end{array}\right| \le 2,$$
$$|c_n| \le 2, n = 1, 2, 3 \cdots . \tag{1.12}$$

**Lemma 1.3.** *(see [18]) Let $p(z) \in \mathrm{P}$. Then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1|^2}{2},$$
$$|c_{n+k} - \mu c_n c_k| < 2 \qquad 0 \le \mu \le 1, \tag{1.13}$$
$$\left|c_{n+2k} - \mu c_n c_k^2\right| \le 2(1 + 2\mu).$$

**Lemma 1.4.** *Let $h(z) = 1 + \sum_{n=2}^{\infty} d_n z^n \prec K(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ in $\mathbb{U}$. If $K(z)$ is convex univalent in $\mathbb{U}$. Then*

$$|d_n| \le |k_1| \ \text{ for } n \ge 1.$$

*We now state and prove the main results of our present investigation.*

**Theorem 1.5.** *If the function $f(z) \in q^*$ and of the form (1.1), then*

$$|a_2| \le \frac{1}{4} + \frac{1}{2}|A - B|,$$
$$|a_3| \le \frac{1}{6} + \frac{1}{2}|A - B|,$$
$$|a_4| \le \frac{23}{96} + \frac{47}{96}|A - B|,$$
$$|a_5| \le \frac{29}{160} + \frac{1}{2}|A - B|, \tag{1.14}$$
$$|a_6| \le 0.6349 + 0.331\,|A - B|,$$
$$|a_7| \le \frac{779}{960} + (0.708)\,|A - B|.$$

*Proof.* Since $f(z) \in q^*$, according to the definition of subordination, then there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\frac{f'(z)}{q(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{1.15}$$

where

$$q(z) \prec \frac{2}{1 + e^{-z}}.$$

We define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{1.16}$$

It is obvious that $p(z) \in \mathcal{P}$ and

$$w(z) = \frac{p(z)+1}{p(z)-1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

On the other hand

$$\begin{aligned}
\frac{2}{1+e^{-w(z)}} = 1 &+ \frac{c_1}{4} z + \left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) z^2 + \left(-\frac{c_1 c_2}{4} + \frac{c_3}{4} + \frac{11 c_1^3}{192}\right) z^3 \\
&+ \left(-\frac{c_2^2}{8} + \frac{11 c_1^2 c_2}{64} - \frac{3 c_1^4}{128} - \frac{c_1 c_3}{4} + \frac{c_4}{4}\right) z^4 \\
&+ \left(-\frac{c_1 c_4}{4} - \frac{c_2 c_3}{4} + \frac{11 c_1^2 c_3}{64} + \frac{11 c_1 c_2^2}{64} - \frac{3 c_1^3 c_2}{32} + \frac{61 c_1^5}{7680} + \frac{c_5}{4}\right) z^5 \\
&+ \left(-\frac{c_1 c_5}{4} - \frac{c_2 c_4}{4} + \frac{11 c_1^2 c_4}{64} - \frac{c_3^2}{8} - \frac{3 c_1^3 c_3}{32} - \frac{9 c_1^2 c_2^2}{64} + \frac{61 c_1^4 c_2}{1536} \right. \\
&\left. + \frac{11 c_2^3}{192} - \frac{5 c_1^6}{3072} + \frac{11 c_1 c_2 c_3}{32} + \frac{c_6}{4}\right) z^6.
\end{aligned} \tag{1.17}$$

Let us put

$$1 + \sum_{n=2}^{\infty} d_n z^n = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{1.18}$$

Then, according to Lemma 1.4, we get

$$|d_n| \leq A - B.$$

Using (1.17) and (1.18), we achieve

$$
\begin{aligned}
&\frac{2}{1+e^{-w(z)}}\,\frac{1+Aw(z)}{1+Bw(z)} \\
&= 1+\left(\frac{c_1}{4}+d_1\right)z+\left(\frac{c_2}{4}-\frac{c_1^2}{8}+\frac{c_1d_1}{4}+d_2\right)z^2+\left(-\frac{c_1c_2}{4}+\frac{c_3}{4}+\frac{11c_1^3}{192}\right. \\
&\left.+d_1\left(\frac{c_2}{4}-\frac{c_1^2}{8}\right)+\frac{c_1d_2}{4}+d_3\right)z^3+\left(-\frac{c_2^2}{8}+\frac{11c_1^2c_2}{64}-\frac{3c_1^4}{128}-\frac{c_1c_3}{4}+\frac{c_4}{4}\right. \\
&\left.+d_1\left(-\frac{c_1c_2}{4}+\frac{c_3}{4}+\frac{11c_1^3}{192}\right)+d_2\left(\frac{c_2}{4}-\frac{c_1^2}{8}\right)+\frac{c_1d_3}{4}+\frac{d_4}{1}\right)z^4+ \\
&\left(-\frac{c_1c_4}{4}-\frac{c_2c_3}{4}+\frac{11c_1^2c_3}{64}+\frac{11c_1c_2^2}{64}-\frac{3c_1^3c_2}{32}+\frac{61c_1^5}{7680}+\frac{c_5}{4}\right. \qquad (1.19) \\
&+d_1\left(-\frac{c_2^2}{8}+\frac{11c_1^2c_2}{64}-\frac{3c_1^4}{128}-\frac{c_1c_3}{4}+\frac{c_4}{4}\right)+d_2\left(-\frac{c_1c_2}{4}+\frac{c_3}{4}+\frac{11c_1^3}{192}\right) \\
&\left.+d_3\left(\frac{c_2}{4}-\frac{c_1^2}{8}\right)+\frac{c_1d_4}{4}+d_5\right)z^5+\left(-\frac{c_1c_5}{5}-\frac{c_2c_4}{5}+\frac{11c_1^2c_4}{64}-\frac{c_3^2}{8}\right. \\
&-\frac{3c_1^3c_3}{32}-\frac{9c_1^2c_2^2}{64}+\frac{64c_1^4c_2}{1536}+\frac{11c_2^3}{192}-\frac{5c_1^6}{3072}+\frac{11c_1c_2c_3}{32}+\frac{c_6}{4} \\
&+d_1\left(-\frac{c_1c_5}{4}-\frac{c_2c_4}{4}+\frac{11c_1^2c_4}{64}-\frac{c_3^2}{8}-\frac{3c_1^3c_3}{32}-\frac{9c_1^2c_2^2}{64}+\frac{61c_1^4c_2}{1536}\right)+ \\
&d_2\left(-\frac{c_2^2}{8}+\frac{11c_1^2c_2}{64}-\frac{3c_1^4}{128}-\frac{c_1c_3}{4}+\frac{c_4}{4}\right)+d_3\left(-\frac{c_1c_2}{4}+\frac{c_3}{4}+\frac{11c_1^3}{192}\right) \\
&\left.+d_4\left(\frac{c_2}{4}-\frac{c_1^2}{8}\right)+\frac{c_1d_5}{4}+d_6\right)z^6.
\end{aligned}
$$

Also

$$f'(z) = 1 + \sum_{n=2}^{\infty} na_n z^{n-1} = 1 + 2a_2 z^1 + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots . \qquad (1.20)$$

When the coefficients of $z, z^2, z^3$ are compared between the equations (1.20) and (1.19), we get

$$a_2 = \frac{c_1}{8} + \frac{d_1}{2}, \tag{1.21}$$

$$a_3 = \frac{1}{3}\left(\frac{c_2}{4} - \frac{c_1^2}{8} + \frac{c_1 d_1}{4} + d_2\right), \tag{1.22}$$

$$a_4 = \frac{1}{4}\left(-\frac{c_1c_2}{4} + \frac{c_3}{4} + \frac{11c_1^3}{192} + d_1\left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) + \frac{c_1 d_2}{4} + d_3\right) \tag{1.23}$$

$$a_5 = \frac{1}{5}\left(-\frac{c_2^2}{8} + \frac{11c_1^2c_2}{64} - \frac{3c_1^4}{128} - \frac{c_1c_3}{4} + \frac{c_4}{4} + d_1\left(-\frac{c_1c_2}{4} + \frac{c_3}{4} + \frac{11c_1^3}{192}\right)\right.$$
$$\left. + d_2\left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) + \frac{c_1d_3}{4} + d_4\right) \tag{1.24}$$

$$a_6 = \frac{1}{6}\left(-\frac{c_1c_4}{4} - \frac{c_2c_3}{4} + \frac{11c_1^2c_3}{64} + \frac{11c_1c_2^2}{64} - \frac{3c_1^3c_2}{32} + \frac{61c_1^5}{7680} + \frac{c_5}{4}\right. +$$
$$\frac{d_1}{6}\left(-\frac{c_2^2}{8} + \frac{11c_1^2c_2}{64} - \frac{3c_1^4}{128} - \frac{c_1c_3}{4} + \frac{c_4}{4}\right) + \frac{d_2}{6}\left(-\frac{c_1c_2}{4} + \frac{c_3}{4} + \frac{11c_1^3}{192}\right)$$
$$\left. + \frac{d_3}{6}\left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) + \frac{c_1d_4}{24} + \frac{d_5}{6}\right), \tag{1.25}$$

$$a_7 = \frac{1}{7}\left(-\frac{c_1c_5}{4} - \frac{c_2c_4}{4} + \frac{11c_1^2c_4}{64} - \frac{c_3^2}{8} - \frac{3c_1^3c_3}{32} - \frac{9c_1^2c_2^2}{64} + \frac{61c_1^4c_2}{1536} + \frac{11c_2^3}{192}\right.$$
$$- \frac{5c_1^6}{3072} + \frac{11c_1c_2c_3}{32} + \frac{c_6}{4} + d_1\left(-\frac{c_1c_5}{4} - \frac{c_2c_4}{4} + \frac{11c_1^2c_4}{64} - \frac{c_3^2}{8} - \frac{3c_1^3c_3}{32}\right.$$
$$\left. - \frac{96c_1^2c_2^2}{64} + \frac{61c_1^4c_2}{1536}\right) + d_2\left(-\frac{c_2^2}{8} + \frac{11c_1^2c_2}{64} - \frac{3c_1^4}{128} - \frac{c_1c_3}{4} + \frac{c_4}{4}\right)$$
$$\left. + d_3\left(-\frac{c_1c_2}{4} + \frac{c_3}{4} + \frac{11c_1^3}{192}\right) + d_4\left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) + \frac{c_1d_5}{4} + \frac{d_6}{1}\right). \tag{1.26}$$

Using Lemma 1.2 and Lemma 1.4, we are easily able to obtain

$$|a_2| \leq \left|\frac{c_1}{8} + \frac{d_1}{2}\right| \leq \frac{c}{8} + \frac{|A-B|}{2} \leq \frac{1}{4} + \frac{|A-B|}{2}. \tag{1.27}$$

$$|a_3| = \left|\frac{c_2}{12} - \frac{c_1^2}{24} + \frac{c_1d_1}{12} + \frac{d_2}{3}\right| = \left|\frac{1}{12}\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{c_1d_1}{12} + \frac{d_2}{3}\right|$$

$+ \quad .$

Using Lemma 1.3 and Lemma 1.4, and suppose that $c_1 = c \in [0, 2]$. Also, if we apply the triangle inequality to the equation above, we get

$$|a_3| \leq \frac{1}{12}\left(2 - \frac{c^2}{2}\right) + \frac{A-B}{6} + \frac{A-B}{3} = \frac{1}{6} - \frac{c^2}{24} + \frac{1}{2}|A-B| .$$

Assume that

$$|a_3| \leq G(c) + \frac{1}{2}|A-B| ,$$

where

$$G(c) = \frac{1}{6} - \frac{c^2}{24},\quad G'(c) = -\frac{c}{12} \le 0$$

As a result, $G(c)$ has a maximum value at $c = 0$. Hence, we have

$$|a_3| \le \frac{1}{6} + \frac{1}{2}|A - B|. \tag{1.28}$$

$$|a_4| = -\frac{c_1c_2}{16} + \frac{c_3}{16} + \frac{11c_1^3}{768} + \frac{d_1}{4}\left(\frac{c_2}{4} - \frac{c_1^2}{8}\right) + \frac{c_1d_2}{16} + \frac{d_3}{4}$$

$$= \frac{[c_3 - c_1c_2]}{16} - \frac{11c_1^3}{768} + \frac{d_1}{4}\frac{[c_2 - c_1^2/2]}{4} + \frac{c_1d_2}{16} + \frac{d_3}{4}.$$

Let $c_1 = c$, $c \in [0, 2]$; by using Lemma 1.3, and Lemma 1.4, we get

$$|a_4| \le H(c) + |A - B|\, M(c). \tag{1.29}$$

Where

$$H(c) = \frac{1}{8} + \frac{11c^3}{768},\quad M(c) = \frac{11}{24} + \frac{c}{16} - \frac{c^2}{32}$$

When we set

$$M'(c) = \frac{1}{16} - \frac{c}{16} = 0,$$

we get $c = 1$, which is the only root of $M'(c) = 0$ in the interval $[0, 2]$ Also $M''(1) < 0$ and $H'(c) > 0$ in the interval $[0, 2]$. As a result, we may be able to write

$$H(2) \le \frac{23}{96} \text{ and } M(1) \le \frac{47}{96}. \tag{1.30}$$

Using (1.30) in (1.29), we achieve

$$|a_4| \le \frac{23}{96} + \frac{47}{96}|A - B|. \tag{1.31}$$

Now

$$|a_5| = \frac{1}{5}\frac{[c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + c_1^4]}{8} + \frac{13c_1^2}{64}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_4}{4}$$
$$+ \frac{d_1}{5}\frac{(c_3 - c_1c_2)}{4} + \frac{11c_1^3}{192} + \frac{d_2}{5}\frac{[c_2 - c_1^2/2]}{4} + \frac{c_1d_3}{20} + \frac{d_4}{5}.$$

Let $c_1 = c$, $c \in [0, 2]$ according to Lemma 1.2, Lemma 1.3 and Lemma 1.4, we achieve

$$|a_5| \le H(c) + |A - B|\, M(c). \tag{1.32}$$

Where

$$H(c) = \frac{3}{20} + \frac{13c^2}{160} - \frac{13c^4}{640},\quad M(c) = \frac{2}{5} + \frac{c}{20} - \frac{c^2}{40} - \frac{11c^3}{960}.$$

Setting $H'(c) = 0$ and $M'(c) = 0$, we get

$$c = \pm\sqrt{2}, c = 1.528,$$

respectively. So for $c = \sqrt{2}$, we achieve

$$H''\left(\sqrt{2}\right) = -0.2161454995 < 0.$$

As a result, at $c = \sqrt{2}$, the function $H(c)$ can obtain the maximum value

$$H\left(\sqrt{2}\right) \le \frac{29}{160}. \tag{1.33}$$

Also, the function $M(c)$ can get the maximum value at $c = 0$

$$M(0) \le \frac{1}{2}, \tag{1.34}$$

using (1.33) and (1.34) in (1.32), we achieve

$$|a_5| \le \frac{29}{160} + \frac{1}{2}|A - B|. \tag{1.35}$$

$$|a_6| = \frac{1}{6}\left| \frac{[c_5 - c_2c_3]}{4} - \frac{61c_1^3[c_2 - c_1^2/2]}{3840} + \frac{11c_1c_2[c_2 - c_1^2/2]}{64} + \frac{31c_1^3c_2}{3840} - c_1\left(c_4 - \frac{13c_1c_3}{16}\right) + d_1\frac{[c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4]}{8} + \frac{c_4}{8} + \frac{13c_1^2}{64}\left(c_2 - \frac{c_1^2}{2}\right) + d_2\left(-\frac{c_1[c_2 - c_1^2/2]}{4} + \frac{(c_3 - c_1c_2)}{4} + \frac{11c_1^3}{192}\right) + d_3\frac{[c_2 - c_1^2/2]}{4} + \frac{c_1d_4}{4} + \frac{d_5}{1}\right|.$$

Let $c_1 = c$, $c \in [0, 2]$, by using Lemma 1.2, Lemma 1.3 and Lemma 1.4, we get

$$|a_6| \le \frac{1}{6}\left[\frac{1}{2} + \frac{61c^3}{3840}\left(2 - \frac{c^2}{2}\right) + \frac{31c^3}{1920} + \frac{11c}{32}\left(2 - \frac{c^2}{2}\right) + \frac{c}{2}\right] + \frac{|A - B|}{6}\left(\frac{1}{2} + \frac{13c^2}{32} - \frac{13c^4}{128}\right) + \frac{|A - B|}{6}\left(\frac{1}{2} - \frac{11c^3}{192}\right) + \frac{|A - B|}{6}\frac{[2 - c^2/2]}{4} + \frac{|A - B|\,c}{24} + \frac{|A - B|}{6}.$$

After a few easy computations, we arrive at

$$|a_6| \le K_1(c) + K_2(c)\,|A - B|. \tag{1.36}$$

Where

$$K_1(c) = \frac{5}{12} + \frac{19c}{96} - \frac{119c^3}{5760} - \frac{61c^5}{46080},$$

$$K_2(c) = \frac{5}{18} + \frac{c}{24} + \frac{3c^2}{64} - \frac{13c^4}{128},$$

When we set

$$K_1'(c) = \frac{19}{96} - \frac{119c^2}{1920} - \frac{61c^4}{9216} = 0,$$

we get $c = 1.587$, which is the only root of $K_1'(c) = 0$ in the interval $[0, 2]$. Also

$$K_1''(1.587) = -0.239 < 0.$$

Obviously, we come across

$$K_1\,(1.587) \leq 0.6349. \qquad (1.37)$$

When we set

$$K_2(c) = \frac{1}{24} + \frac{3c}{32} + \frac{11c^2}{64} - \frac{13c^3}{32} = 0,$$

we get $c = 0.8149$, which is the only root of $K_2(c) = 0$, in the interval $[0, 2]$. Also

$$K_2''(0.8149) = -0.42 < 0.$$

Obviously, we come across

$$K_2(0.8149) \leq 0.331. \tag{1.38}$$

Using (1.37) and (1.38) in (1.36), we get

$$|a_6| \leq 0.6349 + 0.331\,|A - B|.$$

$$|a_7| = \frac{1}{1344}\left| c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6 \right| + \frac{3c_1^2\,[c_4 - c_2^2]}{112} + [c_3 - c_1c_2]\left(\frac{3c_1}{56}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{29c_1^3}{672}\right)$$

$$- \frac{3c_1^4}{448}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{31c_1^6}{7168} + \frac{c_2^2}{112}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{7\,[c_6 - 202/245\,c_2c_4]}{192}$$

$$+ \frac{101c_1c_5}{3360} - \frac{25c_3^2}{1344} + \frac{27c_1^2c_2^2}{448} + \frac{d_1}{7}\left(\frac{[c_5 - c_2c_3]}{4} - \frac{61c_1^3\,[c_2 - c_1^2/2]}{3840} + \frac{11c_1c_2\,[c_2 - c_1^2/2]}{64} + \frac{31c_1^3c_2}{3840} - \frac{c_1}{4}\left(c_4 - \frac{13c_1c_3}{16}\right)\right)$$

$$+ \frac{d_2}{7}\left(\frac{[c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4]}{8} + \frac{13c_1^2}{64}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_4}{8}\right) \tag{1.39}$$

$$+ \frac{d_3}{7}\left(\frac{(c_3 - c_1c_2)}{4} + \frac{11c_1^3}{192}\right) + \frac{d_4}{7}\,\frac{[c_2 - c_1^2/2]}{4} + \frac{c_1d_5}{28} + \frac{d_6}{7}.$$

By taking $c_1 = c$, $c \in [0, 2]$, along with the use of Lemmas 1.2, Lemma 1.3 and Lemma 1.4, we achieve

$$|a_7| \leq \frac{221}{1344} + \frac{281c}{1680} + \frac{c^2}{112} + \frac{3c^4}{224} + \frac{c^6}{1024}$$

$$+ \frac{|A - B|}{7}\left(\frac{1}{2} + \frac{61c^3}{3840}\left(2 - \frac{c^2}{2}\right) + \frac{31c^3}{1920} + \frac{11c}{32}\left(2 - \frac{c^2}{2}\right) + \frac{c}{2}\right)$$

$$+ \frac{|A - B|}{7}\left(\frac{1}{2} + \frac{13c^2}{64}\left(2 - \frac{c^2}{2}\right)\right) + \frac{|A - B|}{7}\left(\frac{1}{2} + \frac{11c^3}{192}\right)$$

$$+ \frac{|A - B|}{\phantom{7}}\,\frac{[2 - c^2/2]}{\phantom{4}} + \frac{|A - B|\,c}{\phantom{28}} + \frac{|A - B|}{\phantom{7}}.$$

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Obviously

$$|a_7| \leq K_3(c) + |A - B|\, K_4(c)\,. \qquad (1.40)$$

Where

$$K_3(c) = \frac{221}{1344} + \frac{281c}{1680} + \frac{c^2}{112} + \frac{3c^4}{224} + \frac{c^6}{1024},$$

$$K_4(c) = -\frac{3}{7} + \frac{23c}{112} + \frac{9c^2}{224} - \frac{c^3}{105} - \frac{13c^4}{896} - \frac{61c^5}{53760},$$

$K_3(c)$ is an increasing function, and its maximum value is given by

$$K_3(c) = K_3(2) \leq \frac{779}{960}. \tag{1.41}$$

When we set

$$K_4'(c) = \frac{23}{112} + \frac{9c}{112} - \frac{c^2}{35} - \frac{13c^3}{224} - \frac{61c^4}{10752} = 0.$$

We get $c = 1.5738$ which is the only root of $K_4(c) = 0$, in the interval $[0, 2]$, also

$$K_4''(1.5738) = -0.539 < 0.$$

As a result, $K_4(c)$ reaches its maximum value at $c = 1.1.5738$

$$K_4(1.5738) \leq 0.708. \tag{1.42}$$

Using (1.41) and (1.42) in (1.40), we get

$$|a_7| \leq \frac{779}{960} + (0.708)\,|A - B|.$$

Hence the proof is completed.

**Theorem 1.6.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6} + \frac{3}{4}|A - B| + \left(\frac{|A - B|}{2}\right)^2. \tag{1.43}$$

*Proof*. From (1.21) and (1.22), we have

$$a_3 - a_2^2 = \frac{c_2}{12} - \frac{11c_1^2}{192} + d_1\frac{5c_1}{24} - d_1^2\frac{c_1}{4} + \frac{d_2}{3},$$

$$= \frac{[c_2 - c_1^2/2]}{12} - \frac{c_1^2}{64} + d_1\frac{5c_1}{24} + \frac{d_2}{3} - \frac{d_1^2}{4}.$$

Using Lemma 1.2, Lemma 1.3 and Lemma 1.4, and putting $c_1 = c$, $c \in [0, 2]$, we get

$$\left|a_3 - a_2^2\right| \leq \frac{[2 - c^2/2]}{12} + \frac{c_1^2}{64} + |A - B|\frac{5c}{24} + \frac{|A - B|}{3} + \left(\frac{|A - B|}{2}\right)^2$$

$$= \frac{1}{6} - \frac{5c^2}{192} + |A - B|\left(\frac{1}{3} + \frac{5c}{24}\right) + \left(\frac{|A - B|}{2}\right)^2.$$

We can express the above inequality as follows:

$$\left|a_3 - a_2^2\right| \leq K_5(c) + K_6(c)\,|A - B| + \left(\frac{|A - B|}{2}\right)^2, \tag{1.44}$$

where

1 5$c^2$ 1 5$c$

$$K_5(c) = \frac{}{6} - \frac{}{192} \text{ and } K_6(c) = \frac{}{3} + \frac{}{24}.$$

$$K_5'(c) = -\frac{5c}{96} < 0.$$

$K_5(c)$ is a decreasing function that reaches maximum value at $c = 0$, while $K_6(c)$ is an increasing function that reaches maximum value at $c = 2$. Hence

$$K_5(c) = K_5(0) \le \frac{1}{6}, \text{ and } K_6(c) = K_6(2) \le \frac{3}{4}. \tag{1.45}$$

We can now use (1.45) in (1.44) to get

$$\left|a_3 - a_2^2\right| \le \frac{1}{6} + \frac{3}{4}|A-B| + \frac{|A-B|^2}{2}.$$

Hence proved the theorem 1.6.

**Theorem 1.7.** *If the function $f(z) \in q^*$ and of the form* (1.1), *then we have*

$$|a_2a_3 - a_4| \le 0.278 + 0.184\,|A-B| + \frac{7}{12}|A-B|^2. \tag{1.46}$$

*Proof*. From (1.21), (1.22) and (1.23), we have

$$|a_2a_3 - a_4| = -\frac{7c_1\left[c_2 - c_1^2/2\right]}{96} + \frac{c_3}{16} - \frac{13c_1^3}{768} + d_1\left(\frac{5\left[c_2 - c_1^2/2\right]}{48} - \frac{c_1^2}{96}\right)$$

$$+ \frac{1}{6}d_1d_2 + \frac{1}{4}d_3d_2 + \frac{c_1}{24}d_1^2 + \frac{c_1}{16}d_2^2 - \frac{c_1}{24}d_2^2.$$

Putting $c_1 = c$, $c \in [0, 2]$, and the use of the Lemmas 1.3 and Lemma 1.4 provide us with the following inequality

$$|a_2a_3 - a_4| \le \frac{7c\left[2 - c^2/2\right]}{96} + \frac{13c^3}{768} + \frac{2}{16} + |A-B|\left(\frac{5\left[2 - c^2/2\right]}{48} + \frac{c^2}{96}\right)$$

$$+ \frac{c}{24}|A-B| + |A-B|^2\,\frac{5+c}{12}.$$

We get the following result after some easy calculation

$$|a_2a_3 - a_4| \le \frac{1}{8} + \frac{7c}{48} - \frac{5c^3}{256} + |A-B|\left(\frac{5}{24} + \frac{c}{24} - \frac{c^2}{24}\right) + |A-B|^2\,\frac{5+c}{12}.$$

Now, we will rewrite the above inequality as follows

$$|a_2a_3 - a_4| \le K_7(c) + |A-B|\,K_8(c) + |A-B|^2\,K_9(c). \tag{1.47}$$

Where

$$K_7(c) = \frac{1}{8} + \frac{7c}{48} - \frac{5c^3}{256}, \tag{1.48}$$

$$K_8(c) = \frac{5}{24} + \frac{c}{24} - \frac{c^2}{24}, \tag{1.49}$$

$$K_9(c) = \frac{5+c}{12}. \tag{1.50}$$

When we set $K_7'(c) = 0$, we get $c = \frac{4\sqrt{35}}{15}$. Then, of course, we will be able to write

$$K_7''(c) = \ldots 15c$$

$$K_7(c) = -\frac{}{128} < 0.$$

As a result, the function $K_7(c)$ reaches its maximum value at $c = r = \frac{4\sqrt{35}}{15}$, also which is

$$K_7(c) = K_7\left(\frac{4\sqrt{35}}{15}\right) \leq 0.278. \quad (1.51)$$

From equation (1.49), we have

$$K_8'(c) = \frac{1}{24} - \frac{c}{12} \leq 0.$$

As a result, the function $K_8(c)$ reaches its maximum value at $c = \sqrt{2}$, also

$$K_8\left(\sqrt{2}\right) \leq 0.184. \quad (1.52)$$

And

$$K_9(c) = \frac{5}{12} + \frac{c}{12},$$

which is an increasing function in the interval $[0, 2]$, so

$$K_9(c) = K_9(2) \leq \frac{7}{12}. \quad (1.53)$$

Now putting (1.51), (1.52) and (1.53) in (1.47), we get

$$|a_2a_3 - a_4| \leq 0.278 + 0.184\,|A - B| + \frac{7}{12}|A - B|^2.$$

The proof of the theorem 1.7 is completed.

**Theorem 1.8.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{553}{4992} + 0.131526\,|A - B| + \frac{25}{144}|A - B|^2 + \frac{3}{16}|A - B|^3. \quad (1.54)$$

*Proof.* From (1.21), (1.22) and (1.23), we have

$$a_2a_4 - a_3^2 = \frac{c_1^2[c_2 - c_1^2/2]}{1152} + \frac{c_2^2}{144} - \frac{c_1c_3}{128} + \frac{1}{18\,432} + d_1d_2\left\{\frac{c_1}{18}\right\}$$
$$+d_1\left(\frac{29c_1^2[c_2 - c_1^2/2]}{1152} + \frac{c_3}{32} - \frac{13c_1^3}{1536}\right) + d_2\frac{[c_2 - c_1^2/2]}{18}$$
$$+d_1^2\left\{\frac{1}{32}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^2}{144}\right\} + d_2^2\left(\frac{1}{9} - \frac{c_1^2}{128}\right) + d_1d_2^2\frac{c_1}{32}$$
$$+d_3d_2\left(\frac{-c_1}{32} - \frac{5c}{576}\right) + \frac{d_3d_1d_2}{8}.$$

Using triangle inequality, and result of Lemma 1.3 and Lemma 1.4, and also let $c_1 = c, c \in [0, 2]$, we get the following inequality

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{36} + \frac{c}{36} + \frac{c^2}{576} + \frac{c^4}{18432} + |A - B|\left(\frac{1}{9} + \frac{29c}{576} + \frac{c^2}{36} + \frac{19c^3}{4608}\right.$$

$$+|A-B|^2\left(\frac{25}{144}+\frac{c}{32}+\frac{c^2}{1152}\right)+|A-B|^3\left(\frac{1}{8}+\frac{c}{32}\right).$$

Suppose that

$$a_2a_4 - a_3^2 \le K_{10}(c) + |A-B|\,K_{11}(c) + |A-B|^2 K_{12}(c) + |A-B|^3 K_{13}(c), \tag{1.55}$$

where

$$K_{10}(c) = \frac{1}{36} + \frac{c}{36} + \frac{c^2}{576} - \frac{c^4}{18432},$$
$$K_{11}(c) = \frac{1}{9} + \frac{29c}{576} - \frac{c^2}{36} - \frac{19c^3}{4608},$$
$$K_{12}(c) = \frac{25}{144} + \frac{c}{32} - \frac{c^2}{1152},$$
$$K_{13}(c) = \frac{1}{8} + \frac{c}{32}.$$

Clearly the function $K_{10}(c)$ is increasing function and reaches its maximum value at $c = 2$. Hence

$$K_{10}(2) \le \frac{553}{4992}. \tag{1.56}$$

The first and second order derivatives of $K_{11}(c)$ are given by

$$K'_{11}(c) = \frac{29}{576} - \frac{c}{18} - \frac{19c^2}{768},$$
$$K''_{11}(c) = -\frac{1}{18} - \frac{19c}{384} < 0.$$

When we set $K'_{11}(c) = 0$, we get $c = 0.77135$ which is the only root in the interval [0, 2], obviously, we find

$$K_{11}(0.77135) \le 0.131526. \tag{1.57}$$

We verified that both $K_{12}(c)$ and $K_{13}(c)$ are increasing functions on [0, 2]. Hence

$$K_{12}(2) \le \frac{67}{288} \text{ and } K_{13}(2) \le \frac{3}{16}. \tag{1.58}$$

Now using (1.56), (1.57) and (1.58) in (1.55). Obviously, we find

$$\left|a_2a_4 - a_3^2\right| \le \frac{553}{4992} + 0.131526\,|A-B| + \frac{25}{144}\,|A-B|^2 + \frac{3}{16}\,|A-B|^3.$$

Hence complete the proof of theorem 1.8.

**Theorem 1.9.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

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$$|a_2 a_5 - a_3 a_4| \le \frac{}{2880} + \frac{}{2880}|A - B| + 0.327\,|A - B| \qquad + \frac{}{16}|A - B| \quad . \qquad (1.59)$$

*Proof*. From (1.21), (1.22), (1.23) and (1.24) we have

$$|a_2a_5 - a_3a_4| = \frac{7c_1^2\left(c_3 - \frac{23c_1c_2}{168}\right)}{1920} + \frac{c_3 - \frac{2c_2c_1}{5}}{192} - \frac{c_1c_4}{480} - \frac{c_1^5}{92160}$$

$$+d_1\left(\frac{7[c_2 - c_1^2/2]}{960} + \frac{3c_1^2[c_2 - c_1^2/2]}{320} - \frac{[c_4 - 24c_1c_3/25]}{40}\right.$$

$$+\frac{157c_1^4}{46080} + d_2\left(\frac{-13c_1[c_2 - c_1^2/2]}{480} + \frac{c_3}{48} - \frac{13c_1^3}{2304}\right)$$

$$+d_1^2\left(\frac{-19c_2[c_2 - c_1^2/2]}{960} + \frac{c_3}{40} - \frac{13c_1^3}{1920}\right) + d_1d_2^2\frac{c_1^2c_2}{192}$$

$$+d_2^3\left\{\frac{c_1}{48}\right\} + d_2d_3\frac{[c_2 - c_1^2/2]}{48} + d_2^2\frac{c_1[c_2 - c_1^2/2]}{192}$$

$$+d_1d_2d_3\left\{\frac{c_1}{48}\right\} + \frac{d_2^2d_3}{12} + d_1d_2\frac{11[c_2 - c_1^2/2]}{240}$$

$$+d_4\frac{-c_1}{40} + d_1d_4\frac{1}{10} + d_1d_3\left\{\frac{c_1}{40}\right\}.$$

Using the Lemma 1.2, Lemma 1.3, and Lemma 1.4 and let $c_1 = c, c \in [0, 2]$, we obtain

$$|a_2a_5 - a_3a_4| \leq \frac{1}{48} + \frac{c}{80} + \frac{7c^2}{960} + \frac{c^5}{92160}$$

$$+|A - B|\left(\frac{29}{240} + \frac{19c}{240} + \frac{17c^2}{960} - \frac{91c^3}{11520} - \frac{59c^4}{46080}\right)$$

$$+|A - B|^2\left(\frac{17}{60} + \frac{31c}{40} - \frac{c^2}{30} - \frac{11c^3}{1920}\right)$$

$$+|A - B|^3\left(\frac{1}{12} + \frac{c}{24} + \frac{c^2}{192}\right).$$

Assume that

$$|a_2a_5 - a_3a_4| \leq K_{14}(c) + |A - B|\,K_{15}(c) + |A - B|^2 K_{16}(c) + |A - B|^3 EK_{17}(c), \tag{1.60}$$

where

$$K_{14}(c) = \frac{1}{48} + \frac{c}{80} + \frac{7c^2}{960} + \frac{c^5}{92160},$$

$$\frac{29}{} \quad \frac{19c}{} \quad \frac{17c^2}{} \quad \frac{91c^3}{} \quad \frac{59c^4}{}$$

$$K_{15}(c) = \frac{}{240} + \frac{}{240} + \frac{}{960} - \frac{}{11520} - \frac{}{46080},$$

$$K_{16}(c) = \frac{17}{60} + \frac{31c}{40} - \frac{c^2}{30} - \frac{11c^3}{1920},$$

$$K_{17}(c) = \frac{1}{12} + \frac{c}{24} + \frac{c^2}{192}.$$

The functions $K_{14}(c)$, $K_{15}(c)$ and $K_{17}(c)$ are increasing functions that reach their maximum value at $c = 2$

$$K_{14}(2) \leq \frac{217}{2880},\ K_{15}(2) \leq \frac{767}{2880} \text{ and } K_{17}(2) \leq \frac{3}{16}. \tag{1.61}$$

When we set $K_{16}'(c) = 0$, we get $c = 0.91102$ which is the only root in the interval [0, 2], obviously, we find

$$K_{16}''(c) = -\frac{1}{15} - \frac{11c}{320} < 0,$$

$$K_{16}(0.91102) \leq 0.327. \tag{1.62}$$

Now setting (1.61), and (1.62) in (1.60), we have

$$|a_2a_5 - a_3a_4| \leq \frac{217}{2880} + \frac{767}{2880}|A-B| + 0.327\,|A-B|^2 + \frac{3}{16}|A-B|^3.$$

We complete the proof of theorem 1.9.

**Theorem 1.10.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

$$|a_5 - a_2a_4| \leq 0.245 + \frac{37}{80}|A-B| + \frac{1}{16}|A-B|^2 + \frac{3}{16}|A-B|^3. \tag{1.63}$$

*Proof.* From (1.21), (1.23) and (1.24), we have

$$\begin{aligned}|a_5 - a_2a_4| = &-\frac{c_2[c_2 - c_1^2/2]}{40} + \frac{19c_1^2[c_2 - c_1^2/2]}{640} - \frac{[c_4 - 32c_1c_3/37]}{20} + \frac{257c_1^4}{30720} \\ &+d_1\left(-\frac{57c_1[c_2 - c_1^2/2]}{640} + \frac{13c_3}{160} - \frac{169c_1^3}{7680}\right) + d_2\frac{[c_2 - c_1^2/2]}{20} \\ &+d_3\left(\frac{c_1}{20} - \frac{d_2c_1}{32} + \frac{d_1d_2}{8}\right) + \frac{d_4}{5} + d_1^2\frac{[c_2 - c_1^2/2]}{32} \\ &+d_2^2\left(\frac{d_1c_1}{32} - \frac{c_1^2}{128}\right).\end{aligned}$$

Using triangle inequality and results of Lemma 1.3 and Lemma 1.4, and $c_1 = c \in [0, 2]$, we obtain

$$|a_5 - a_2a_4| \leq \frac{1}{5} + \frac{11c^2}{320} - \frac{199c^4}{30720} + |A-B|\left(\frac{37}{80} + \frac{73c}{320} - \frac{1c^2}{40} - \frac{173c^3}{7680}\right) + |A-B|^2\left(\frac{1}{...} - \frac{c}{...} - \frac{c^2}{...}\right) + |A-B|^3\left(\frac{1}{...} - \frac{c}{...}\right)$$

$$|A-B| \quad 16 + 32 - 128 \quad + |A-B| \quad 8 + 32 \quad .$$

Suppose that

$$|a_5 - a_2a_4| \leq K_{18}(c) + |A-B|\, K_{19}(c) + |A-B|^2\, K_{20}(c) + |A-B|^3\, K_{21}(c), \tag{1.64}$$

where

$$K_{18}(c) = \frac{1}{5} + \frac{11c^2}{320} - \frac{199c^4}{30720},$$

$$K_{19}(c) = \frac{37}{80} + \frac{73c}{320} - \frac{c^2}{40} - \frac{173c^3}{7680},$$

$$K_{20}(c) = \frac{1}{16} + \frac{c}{32} - \frac{c^2}{128},$$

$$K_{21}(c) = \frac{1}{8} + \frac{c}{32}.$$

When we set $K'_{18}(c) = 0$, then we get $c = 1.629$. As a result, we determine the following:

$$K''_{18}(c) = \frac{11}{160} - \frac{199c^3}{2560},$$

$$K''_{18}(1.629) = -0.13753 < 0.$$

Hence, we are able to write

$$K_{18}(1.629) \leq 0.245. \tag{1.65}$$

In the same way, we can deduce that $K_{19}(c)$, $K_{20}(c)$ are a decreasing functions and $K_{21}(c)$ is an increasing function, therefore

$$K_{19}(0) \leq \frac{37}{80}. \tag{1.66}$$

$$K_{20}(0) \leq \frac{1}{16}. \tag{1.67}$$

$$K_{21}(2) \leq \frac{3}{16}. \tag{1.68}$$

Using (1.65), (1.66), (1.67) and (1.68) in (1.64), we get

$$|a_5 - a_2a_4| \leq 0.245 + \frac{37}{80}|A-B| + \frac{1}{16}|A-B|^2 + \frac{3}{16}|A-B|^3.$$

The proof of the theorem 1.10 is now complete.

**Theorem 1.11.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

$$a_3a_5 - a_4^2 \leq \frac{3077}{4608} + 0.136|A-B| + \frac{79}{480}|A-B|^2 + 0.067|A-B|^3 + \frac{5}{64}|A-B|^4. \tag{1.69}$$

*Proof*. From (1.22), (1.23) and (1.24), we have

$$a_3a_5 - a_4^2 = -\frac{c_4}{240}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{3c_1^3\,(c_3 - c_1c_2/3)}{10240} - \frac{7c_1c_2c_3}{1920} + \frac{29c_1^6}{2949120}$$

$$+\frac{c_2^3}{480} + \frac{c_3^2}{256} + d_1d_2\left(-\frac{c_1^3}{576} - \frac{(c_3 - 3c_1c_2/4)}{160}\right) + d_2^4\,\frac{c_1^2}{256}$$

$$+d_1\left(\frac{3c_1^2}{1280}\left(c_3 - \frac{19c_2}{216}\right) + \frac{c_1}{240}\left(c_4 - \frac{3c_2^2}{8}\right) + \frac{7c_2c_3}{1920} - \frac{c_1^5}{36864}\right)$$

$$+d_2\left(\frac{-c_1^2[c_2 - c_1^2/2]}{960} + \frac{c_2\,[c_2 - c_1^2/2]}{240} - \frac{[c_4 - c_1c_3]}{60} - \frac{c_1^2}{240}\right)$$

$$+d_3\,\frac{-c_1}{240}\left(c_2 - \frac{c_1^2}{2}\right) + d_4\,\frac{-[c_2 - c_1^2/2]}{60}$$

$$+d_1^2\left(\frac{-c_1}{240}\left(c_3 - \frac{c_1c_2}{16} - \frac{c_2^2}{256}\right) + \frac{c_1^4}{46\,080}\right) + d_1d_2d_3\,\frac{[c_2 - c_1^2/2]}{32}$$

$$+d_2^2\left(-\frac{1}{60}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{c_1\,(c_3 - c_1c_2)}{128} + \frac{11c_1^4}{46144}\right)$$

$$+d_1d_3\,\frac{-c_1^2}{240} + d_1d_2^2\,\frac{c_1\,[c_2 - c_1^2/2]}{128} + d_1d_4\,\frac{-c_1}{60} + \frac{d_2d_4}{15}$$

$$+d_2d_3\left(\frac{c_1\,(c_3 - c_1c_2)}{128} + \frac{11c_1^3}{1536}\right) + d_2^2d_3\left[\frac{c_1}{32}\right] + \frac{d_2^2d_3^2}{16}.$$

Using triangle inequality and results of Lemma 1.2 and Lemma 1.3, and Lemma 1.4 also $c_1 = c, c \in [0, 2]$, we obtain

$$a_3a_5 - a_4^2 \leq \frac{47}{960} + \frac{7c}{480} - \frac{c^2}{240} + \frac{3c^3}{5120} + \frac{29c^6}{2949120}$$

$$+|A - B|\left(\frac{47}{480} + \frac{c}{30} + \frac{c^2}{384} - \frac{c^3}{480} - \frac{c^4}{480} + \frac{c^5}{36864}\right)$$

$$+|A - B|^2\left(\frac{79}{840} + \frac{53c}{960} - \frac{c^2}{120} + \frac{41c^3}{4608} + \frac{37c^4}{18432}\right)$$

$$+|A - B|^3\left(\frac{1}{16} + \frac{c}{64} - \frac{c^2}{64} - \frac{c^3}{256}\right) + |A - B|^4\left(\frac{1}{16} + \frac{c^2}{256}\right).$$

Assume that

$$a_3a_5 - a^2_4 \leq K_{22}(c) + |A-B|\, K_{23}(c) + |A-B|^2 K_{24}(c) \quad (1.70)$$
$$+ |A-B|^3 K_{25}(c) + |A-B|^4 K_{26}(c) .$$

Where

$$K_{22}(c) = \frac{47}{960} + \frac{7c}{480} - \frac{c^2}{240} + \frac{3c^3}{5120} + \frac{29c^6}{2949120},$$

$$K_{23}(c) = \frac{47}{480} + \frac{c}{30} + \frac{c^2}{384} - \frac{c^3}{480} - \frac{c^4}{480} + \frac{c^5}{36864},$$

$$K_{24}(c) = \frac{79}{840} + \frac{53c}{960} + \frac{c^2}{120} + \frac{41c^3}{4608} + \frac{37c^4}{18432},$$

$$K_{25}(c) = \frac{1}{16} + \frac{c}{64} - \frac{c^2}{64} - \frac{c^3}{256},$$

$$K_{26}(c) = \frac{1}{16} + \frac{c^2}{256}.$$

When we set $K_{23}'(c) = 0$, we get $c = 1.51$ which is the only root of $K_{23}'(c) = 0$ in the interval $[0, 2]$. As a result, we observe

$$K_{23}''(1.51) = -0.362 < 0.$$

Hence,

$$K_{23}(1.51) \leq 0.136. \tag{1.71}$$

We observed that $K_{22}(c)$, $K_{24}(c)$ and $K_{26}(c)$ are increasing functions, allowing us to write the following inequalities

$$K_{22}(2) \leq \frac{3077}{46080}, K_{24}(2) \leq 0.34110. \text{and } K_{26}(2) \leq \frac{5}{64}. \tag{1.72}$$

We can achieve similar results in a similar way

$$K_{25}(0.4305) \leq 0.067. \tag{1.73}$$

Using (1.71), (1.72) and (1.73) in (1.70), we obtain

$$\left|a_3a_5 - a_4^2\right| \leq \frac{3077}{4608} + 0.136\,|A-B| + 0.34110\,|A-B|^2 + 0.067\,|A-B|^3 + \frac{5}{64}\,|A-B|^4.$$

We complete the proof of theorem 1.11.

**Theorem 1.12.** *If the function $f(z) \in q^*$ and of the form (1.1), then we have*

$$\begin{aligned}|H_4(1)| \leq\ & 0.33293 + 1.6543\,|A-B| + 1.7763\,|A-B|^2 \\ & +1.6895\,|A-B|^3 + 1.7584\,|A-B|^4 + 0.49179\,|A-B|^5 \\ & + \frac{22565}{589824}\,|A-B|^6. \end{aligned} \tag{1.74}$$

*Proof.*

$$\begin{aligned} H_4(1) = a_7 & \left\{a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right)\right\} \\ - a_6 & \left\{a_3\left(a_2a_5 - a_3a_4\right) - a_4\left(a_5 - a_2a_4\right) + a_6\left(a_3 - a_2^2\right)\right\} \\ + a_5 & \left\{a_3\left(a_3a_5 - a_4^2\right) - a_5\left(a_5 - a_2a_4\right) + a_6\left(a_4 - a_2a_3\right)\right\} \\ - a_4 & \left\{a_4\left(a_3a_5 - a_4^2\right) - a_5\left(a_2a_5 - a_3a_4\right) + a_6\left(a_4 - a_2a_3\right)\right\}, \end{aligned}$$

so, by applying the triangle inequality, we obtain

$$\begin{aligned} H_4(1) \le\ & |a_7|\,|a_3|\,\left|a_2a_4 - a_3^2\right| + |a_4|\,|a_7|\,|a_4 - a_2a_3| + |a_5|\,|a_7|\,\left|a_3 - a_2^2\right| \\ & + |a_6|\,|a_3|\,|a_2a_5 - a_3a_4| + |a_4|\,|a_6|\,|a_5 - a_2a_4| + |a_6|^2\,\left|a_3 - a_2^2\right| \\ & + |a_5|\,|a_3|\,\left|a_3a_5 - a_4^2\right| + |a_5|^2\,|a_5 - a_2a_4| + |a_5|\,|a_6|\,|a_4 - a_2a_3| \\ & + |a_4|^2\,\left|a_3a_5 - a_4^2\right| + |a_4|\,|a_5|\,|a_2a_5 - a_3a_4| + |a_4|\,|a_6|\,|a_4 - a_2a_3|. \end{aligned} \tag{1.75}$$

Next, substituting Equations (1.14),(1.43),(1.46),(1.54),(1.59), (1.64), (1.69), (1.60) into (1.75), we easily get the desired assertion equation (1.74) .

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